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## Spectral problems for operator matrices

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*Dedicated to the memory of Professor F. V. Atkinson with respect and admiration*

We study spectral properties of  $2 \times 2$  block operator matrices whose entries are unbounded operators between Banach spaces and with domains consisting of vectors satisfying certain relations between their components. We investigate closability in the product space, essential spectra and generation of holomorphic semigroups. Application is given to several models governed by ordinary and partial differential equations, for example containing delays, floating singularities or eigenvalue dependent boundary conditions.

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## 1 Introduction

Let  $X, Y$  be Banach spaces and  $\mathcal{X}$  be the product space  $\mathcal{X} = X \times Y$ . In this paper we consider operators in  $\mathcal{X}$  which, with respect to this decomposition, are formally given by block operator matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}. \quad (1.1)$$

Here we write *formally* since in the applications which we have in mind the operators  $A, B, C, D$  are in general unbounded and then the operator defined by the matrix in (1.1) on  $(\mathcal{D}(A) \cap \mathcal{D}(C)) \times (\mathcal{D}(B) \cap \mathcal{D}(D))$  need not be closed, or the domain of this operator is determined by an additional relation of the form  $\Gamma_X x = \Gamma_Y y$  between the components  $x$  and  $y$  of its elements.

The study of such operator matrices was started in [23], [24] and, independently and under slightly different assumptions, later in [3]. The common tool in these investigations is a Frobenius–Schur factorization for the matrix in (1.1), and hence some conditions for the definition of the operator associated with (1.1) involve the corresponding Schur complements. The situation where the domains of the diagonal operators satisfy  $\mathcal{D}(A) \subset \mathcal{D}(C)$ ,  $\mathcal{D}(D) \subset \mathcal{D}(B)$  was considered in [23]. For example, in the case of differential operators, this means that the orders of  $A$  and  $D$  are not lower than the orders of  $C$  and  $B$ , respectively. In [3] it was assumed that  $\mathcal{D}(A) \subset \mathcal{D}(C)$ ,  $\mathcal{D}(B) \subset \mathcal{D}(D)$ . An example of this second case is the linearized Navier–Stokes equation, but both cases can occur, e.g., in magnetohydrodynamics, population dynamics, damped plate equations with delay,

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etc. We mention that there are also situations where the domains of the off-diagonal operators  $B$  and  $C$  are the smaller ones, e.g., for the Dirac operator, see [19].

A starting point for the present investigation was the paper [8] (which, in turn, was inspired by [14]) on elliptic boundary value problems in some domain  $\Omega$  with boundary conditions on the boundary  $\partial\Omega$  depending linearly on the eigenvalue parameter. It was shown there that this problem can be considered in a natural way in the orthogonal sum of two Hilbert spaces, consisting of functions defined on  $\Omega$  and  $\partial\Omega$ , respectively, together with an operator of the form (1.1). Moreover some Sturm–Liouville problems, for which the differential equation and also the boundary conditions depend on the eigenvalue parameter in a rational way, have a linearization which can be described by such an operator matrix, and the same is true for certain differential equations with delay, see [24]. We mention that examples from [23] and [3] also fit in our framework but we shall not repeat them here.

Motivated by the above examples, we assume in this paper that  $\mathcal{D}(A) \subset \mathcal{D}(C)$ , and that the intersection of the domains of  $B$  and  $D$  is sufficiently large. Moreover, the domain of the operator matrix is defined by an additional relation between the two components of its elements. In comparison with the recent works [5], [6], [22] on matrix operators of the form (1.1) arising from differential equations with delay, we allow more general off-diagonal terms. The main focus in those papers was on semigroup generation. Here we also study spectral properties of the corresponding matrix operators in more detail. Besides the definition of the (closed) operator  $\mathcal{A}$  associated with the operator matrix (1.1) we are interested in the spectrum of  $\mathcal{A}$ , and in particular in its essential spectrum.

The organization of the paper is as follows. In the next section we introduce the assumptions (i–viii) to be imposed on the operator matrix (1.1), and in Section 3 we use these assumptions to define a closed operator  $\mathcal{A}$  associated with (1.1). The essential spectrum of  $\mathcal{A}$  is determined in Section 4, and in Section 5 we introduce conditions which ensure that  $\mathcal{A}$  generates a holomorphic semigroup. In the following three sections we apply the abstract theory of Sections 2–5 to the above mentioned Sturm–Liouville problem (Section 6); to elliptic problems, of a more general form than considered in [8] and arising, e.g., in the study of diffusion processes [26] (Section 7); and to boundary feedback problems (Section 8). Finally, in Section 9 we show that differential equations with delay and abstract observation problems also fall within the class of operator matrices considered here.

In the present paper we do not pay special attention to Hilbert space structure nor to self-adjointness of the operators. These and related questions will be considered elsewhere.

Throughout the paper we denote by  $\mathcal{D}(T)$ ,  $\mathcal{N}(T)$ , and  $\mathcal{R}(T)$  the domain, nullspace, and range of an operator  $T$  acting between Banach spaces.

## 2 Preliminaries

Let  $X, Y, Z$  be Banach spaces. We consider linear operators

$A$  in  $X$ ,  $D$  in  $Y$ ,  $C$  from  $X$  into  $Y$ ,  $B$  from  $Y$  into  $X$ ,  $\Gamma_X$  from  $X$  into  $Z$ ,  $\Gamma_Y$  from  $Y$  into  $Z$ ,

with the following properties (i)–(viii).

(i) *The operator  $A$  is densely defined and closable.*

It follows that  $\mathcal{D}(A)$  equipped with the graph norm

$$\|x\|_A := \|x\| + \|Ax\|, \quad x \in \mathcal{D}(A),$$

can be completed to a Banach space  $X_A$ , which coincides with  $\mathcal{D}(\overline{A})$ , the domain of the closure  $\overline{A}$  of  $A$ , and which is contained in  $X$ .

(ii)  $\mathcal{D}(A) \subset \mathcal{D}(\Gamma_X) \subset X_A$  and  $\Gamma_X$  is bounded as a mapping from  $X_A$  into  $Z$ .

The extension of  $\Gamma_X$  by continuity to  $X_A = \mathcal{D}(\overline{A})$  is denoted by  $\overline{\Gamma}_X$ , which is a bounded operator from  $X_A$  into  $Z$ .

(iii) *The set  $\mathcal{D}(A) \cap \mathcal{N}(\Gamma_X)$  is dense in  $X$  and the resolvent set of the restriction  $A_1 := A|_{\mathcal{D}(A) \cap \mathcal{N}(\Gamma_X)}$  is not empty, i.e.,  $\rho(A_1) \neq \emptyset$ .*

(iv)  $\mathcal{D}(A) \subset \mathcal{D}(C) \subset X_A$  and  $C$  is closable as an operator from  $X_A$  into  $Y$ .

It follows from (iii) that

$$\Gamma_X(\mathcal{D}(A_1)) = \{0\},$$

and that  $A_1$  is a closed operator, whence  $\mathcal{D}(A_1)$  is a closed subspace of  $X_A$ . The closed graph theorem and (iv) imply that for  $\lambda \in \rho(A_1)$  the operator

$$C_\lambda := C(A_1 - \lambda I)^{-1} \quad (2.1)$$

from  $X$  into  $Y$  is bounded.

These assumptions allow  $\mathcal{D}(A)$  to be decomposed as follows; for a more general result see [24].

**Lemma 2.1** *Under assumptions (i)–(iii), for any  $\lambda \in \rho(A_1)$  the following decomposition holds:*

$$\mathcal{D}(A) = \mathcal{D}(A_1) \dot{+} \mathcal{N}(A - \lambda I). \quad (2.2)$$

**Proof.** The sum in (2.2) is contained in  $\mathcal{D}(A)$  and is direct since, by assumption,

$$\mathcal{N}(A - \lambda I) \cap \mathcal{D}(A_1) = \mathcal{N}(A_1 - \lambda I) = \{0\}.$$

Take any  $f \in \mathcal{D}(A)$  and set

$$g := (A_1 - \lambda I)^{-1}(A - \lambda I)f \in \mathcal{D}(A_1).$$

Then  $f - g \in \mathcal{N}(A - \lambda I)$  and

$$f = g + (f - g) \in \mathcal{D}(A_1) \dot{+} \mathcal{N}(A - \lambda I). \quad \square$$

**Lemma 2.2** *Under assumptions (i)–(iii), for any  $\lambda \in \rho(A_1)$  the restriction*

$$\Gamma_\lambda := \Gamma_X|_{\mathcal{N}(A - \lambda I)} \quad (2.3)$$

*is injective and*

$$\mathcal{R}(\Gamma_\lambda) = \Gamma_X(\mathcal{N}(A - \lambda I)) = \Gamma_X(\mathcal{D}(A)) =: Z_1 \quad (2.4)$$

*does not depend on  $\lambda$ .*

**Proof.** For the first statement we observe that

$$\mathcal{N}(A - \lambda I) \cap \mathcal{N}(\Gamma_X) = \mathcal{N}(A_1 - \lambda I).$$

The second statement follows from the fact that, because of (2.2) and  $\Gamma_X(\mathcal{D}(A_1)) = \{0\}$ ,

$$\Gamma_X(\mathcal{N}(A - \lambda I)) = \Gamma_X(\mathcal{D}(A)). \quad \square$$

In the following, for  $\lambda \in \rho(A_1)$  the inverse  $K_\lambda$  of the operator  $\Gamma_\lambda$  in (2.3) will play an important role:

$$K_\lambda := \left( \Gamma_X|_{\mathcal{N}(A - \lambda I)} \right)^{-1} : Z_1 \longrightarrow \mathcal{N}(A - \lambda I) \subset X. \quad (2.5)$$

In other words,  $K_\lambda z = x$  means that  $x \in \mathcal{D}(A)$  and

$$Ax = \lambda x, \quad (2.6)$$

$$\Gamma_X x = z. \quad (2.7)$$

**Lemma 2.3** *For  $\lambda \in \rho(A_1)$  and  $x \in \mathcal{D}(A)$  we have*

$$(A - \lambda I)x = (A_1 - \lambda I)(I - K_\lambda \Gamma_X)x,$$

*and the operator  $I - K_\lambda \Gamma_X$  is the projection from  $\mathcal{D}(A)$  onto  $\mathcal{D}(A_1)$  parallel to  $\mathcal{N}(A - \lambda I)$ .*

**Proof.** Every  $x \in \mathcal{D}(A)$  can be written as

$$x = (I - K_\lambda \Gamma_X)x + K_\lambda \Gamma_X x.$$

The second summand belongs to  $\mathcal{N}(A - \lambda I)$ , the first belongs to  $\mathcal{D}(A_1)$  because  $x' = (I - K_\lambda \Gamma_X)x \in \mathcal{D}(A)$  and

$$\Gamma_X x' = \Gamma_X x - \Gamma_X K_\lambda \Gamma_X x = 0.$$

It remains to apply Lemma 2.1. □

**Lemma 2.4** *If  $\lambda_1, \lambda_2 \in \rho(A_1)$ , then*

$$K_{\lambda_1} - K_{\lambda_2} = (\lambda_1 - \lambda_2)(A_1 - \lambda_1 I)^{-1} K_{\lambda_2}.$$

*If  $K_\lambda$  is closable for at least one  $\lambda \in \rho(A_1)$ , then it is closable for all such  $\lambda$ , and the above relation holds with  $K_{\lambda_j}$  replaced by the closures  $\overline{K}_{\lambda_j}$ ,  $j = 1, 2$ .*

**Proof.** Consider  $z \in Z_1$  and set  $x_j := K_{\lambda_j} z$ ,  $j = 1, 2$ . In view of (2.6) and (2.7), the element  $x := x_1 - x_2$  satisfies the relations

$$\begin{aligned} (A - \lambda_1 I)x &= -(A - \lambda_1 I)x_2 = (\lambda_1 - \lambda_2)x_2, \\ \Gamma_X x &= \Gamma_X x_1 - \Gamma_X x_2 = 0, \end{aligned}$$

whence  $x \in \mathcal{D}(A_1)$  and  $x = (\lambda_1 - \lambda_2)(A_1 - \lambda_1 I)^{-1}x_2$ . This yields the required formula and, in turn, the relation

$$K_{\lambda_1} = (A_1 - \lambda_2 I)(A_1 - \lambda_1 I)^{-1} K_{\lambda_2}.$$

Since the operator  $(A_1 - \lambda_2 I)(A_1 - \lambda_1 I)^{-1}$  is bounded and boundedly invertible,  $K_{\lambda_1}$  is closable if  $K_{\lambda_2}$  is such, in which case their closures  $\overline{K}_{\lambda_j}$ ,  $j = 1, 2$ , satisfy the same relations. □

We also impose the following condition:

(v) *For some  $\lambda_1 \in \rho(A_1)$ , the operator  $K_{\lambda_1}$  is bounded as a mapping from  $Z$  into  $X$ .*

If this condition is satisfied, then it is satisfied for all  $\lambda \in \rho(A_1)$  by Lemma 2.4. Moreover, in this case  $K_\lambda$  can be extended by continuity to the closure  $\overline{Z}_1$  of  $Z_1$  with respect to the norm of  $Z$ ; we denote this extension by  $\overline{K}_\lambda$ . Without loss of generality we assume that  $\overline{Z}_1 = Z$ .

Since for  $x \in \mathcal{N}(A - \lambda I)$  we have  $\|x\|_A = (1 + |\lambda|)\|x\|$ , the operator  $\overline{K}_\lambda$  is also bounded as a mapping from  $\overline{Z}_1$  to  $X_A$ , which implies that

$$\overline{A} \overline{K}_\lambda z = \lambda \overline{K}_\lambda z, \quad \overline{\Gamma}_X \overline{K}_\lambda z = z \quad (z \in \overline{Z}_1).$$

Concerning the operators  $D$  and  $\Gamma_Y$  we assume:

(vi) *The operator  $D$  is densely defined and closed with  $\rho(D) \neq \emptyset$ .*

(vii)  *$\mathcal{D}(\Gamma_Y) \supset \mathcal{D}(D) \cap \mathcal{D}(B)$ , the set*

$$Y_1 := \{y \mid y \in \mathcal{D}(D) \cap \mathcal{D}(B), \Gamma_Y y \in Z_1\} \tag{2.8}$$

*is dense in  $Y$ , and the restriction of  $\Gamma_Y$  to this set is bounded as an operator from  $Y$  into  $Z$ .*

We denote the extension of  $\Gamma_Y|_{Y_1}$  by continuity to all of  $Y$  by  $\overline{\Gamma}_Y^0$ .

Finally, concerning the operator  $B$  we assume:

(viii) *For some  $\lambda \in \rho(A_1)$ , the operator  $(A_1 - \lambda I)^{-1}B$  is closable and its closure  $\overline{(A_1 - \lambda I)^{-1}B}$  is bounded.*

Observe that (viii) and the resolvent identity imply that the operator  $(A_1 - \lambda I)^{-1}B$  is closable for all  $\lambda \in \rho(A_1)$  and, moreover, if  $\lambda_1, \lambda_2 \in \rho(A_1)$  then we have

$$\overline{(A_1 - \lambda_1 I)^{-1}B} - \overline{(A_1 - \lambda_2 I)^{-1}B} = (\lambda_1 - \lambda_2)(A_1 - \lambda_1 I)^{-1}\overline{(A_1 - \lambda_2 I)^{-1}B}. \quad (2.9)$$

Recall the notation (2.1), (2.5). In the next section the operator

$$M_\lambda := D + CK_\lambda\Gamma_Y - C_\lambda B \quad (2.10)$$

in the space  $Y$  will play an important role for  $\lambda \in \rho(A_1)$ . It is defined on the set  $Y_1$ , which is dense in  $Y$  according to (vii). Here we observe that  $\Gamma_Y$  is bounded on this domain by assumption (vii), that  $K_\lambda$  is bounded by assumption (v), and, finally,  $\mathcal{R}(K_\lambda) \subset \mathcal{D}(A) \subset \mathcal{D}(C)$  and  $C_\lambda$  is bounded by assumption (iv). Hence the right side of

$$M_{\lambda_1} - M_\lambda = (\lambda_1 - \lambda)C_{\lambda_1}[K_\lambda\Gamma_Y - (A_1 - \lambda I)^{-1}B] \quad (2.11)$$

is bounded, and assumptions (iv) and (viii) imply that if  $M_\lambda$  is closable as an operator in  $Y$  for some  $\lambda_1 \in \rho(A_1)$  then it is also closable for any  $\lambda \in \rho(A_1)$ . In this case the domain of the closure  $\overline{M}_\lambda$  is independent of  $\lambda \in \rho(A_1)$ ; in fact the difference

$$\overline{M}_{\lambda_1} - \overline{M}_\lambda = (\lambda_1 - \lambda)C_{\lambda_1}[\overline{K}_\lambda\overline{\Gamma}_Y^\circ - \overline{(A_1 - \lambda I)^{-1}B}] \quad (2.12)$$

is a bounded operator.

### 3 The operator $\mathcal{A}_0$ and its closure $\mathcal{A}$

Throughout this and the following two sections we suppose that assumptions (i)–(viii) are satisfied. We introduce the Banach space  $\mathcal{X} := X \times Y$  and define the operator  $\mathcal{A}_0$  in  $\mathcal{X}$  as follows:

$$\mathcal{D}(\mathcal{A}_0) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x \in \mathcal{D}(A), y \in \mathcal{D}(D) \cap \mathcal{D}(B), \Gamma_X x = \Gamma_Y y \right\}, \quad (3.1)$$

$$\mathcal{A}_0 \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} Ax + By \\ Cx + Dy \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{D}(\mathcal{A}_0). \quad (3.2)$$

Our aim is to describe the closure  $\mathcal{A}$  of the operator  $\mathcal{A}_0$  in  $\mathcal{X}$ . We start with the following Frobenius–Schur type factorization of  $\mathcal{A}_0$ .

**Lemma 3.1** *If  $\lambda \in \rho(A_1)$ , then*

$$\mathcal{A}_0 - \lambda \mathcal{I} = \begin{pmatrix} I & 0 \\ C_\lambda & I \end{pmatrix} \begin{pmatrix} A_1 - \lambda I & 0 \\ 0 & M_\lambda - \lambda I \end{pmatrix} \begin{pmatrix} I & -K_\lambda\Gamma_Y + (A_1 - \lambda I)^{-1}B \\ 0 & I \end{pmatrix}.$$

**Proof.** Denote by  $\mathcal{B}_\lambda$  the operator on the right-hand side of the above equality, and suppose that  $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{D}(\mathcal{A}_0)$ , i.e.,  $x \in \mathcal{D}(A)$ ,  $y \in \mathcal{D}(D) \cap \mathcal{D}(B)$ , and  $\Gamma_X x = \Gamma_Y y$ . Then  $x - K_\lambda\Gamma_Y y = (I - K_\lambda\Gamma_X)x$ , and by Lemma 2.3 we have

$$(A_1 - \lambda I)(I - K_\lambda\Gamma_X)x = (A - \lambda I)x.$$

It follows that

$$\begin{aligned} \mathcal{B}_\lambda \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} A_1 - \lambda I & 0 \\ C & M_\lambda - \lambda I \end{pmatrix} \begin{pmatrix} (I - K_\lambda\Gamma_X)x + (A_1 - \lambda I)^{-1}By \\ y \end{pmatrix} \\ &= \begin{pmatrix} (A - \lambda I)x + By \\ C(x - K_\lambda\Gamma_X x + (A_1 - \lambda I)^{-1}By) + (M_\lambda - \lambda I)y \end{pmatrix} \\ &= \begin{pmatrix} (A - \lambda I)x + By \\ Cx + (D - \lambda I)y \end{pmatrix} = (\mathcal{A}_0 - \lambda \mathcal{I}) \begin{pmatrix} x \\ y \end{pmatrix}, \end{aligned}$$

hence  $\mathcal{A}_0 - \lambda \mathcal{I} \subset \mathcal{B}_\lambda$ .

It remains to show that  $\mathcal{D}(\mathcal{B}_\lambda) \subset \mathcal{D}(\mathcal{A}_0)$ . Obviously,  $\mathcal{D}(\mathcal{B}_\lambda)$  coincides with the set

$$\begin{pmatrix} I & K_\lambda \Gamma_Y - (A_1 - \lambda)^{-1} B \\ 0 & I \end{pmatrix} \mathcal{D}(A_1) \times \mathcal{D}(M_\lambda),$$

that is, it consists of the elements of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' + K_\lambda \Gamma_Y y - (A_1 - \lambda I)^{-1} B y \\ y \end{pmatrix},$$

where  $x'$  and  $y$  run through  $\mathcal{D}(A_1) = \mathcal{D}(A) \cap \mathcal{N}(\Gamma_X)$  and  $\mathcal{D}(M_\lambda) = Y_1$ , see (2.8), respectively. Therefore  $x \in \mathcal{D}(A)$ ,  $y \in \mathcal{D}(D) \cap \mathcal{D}(B)$ , and  $\Gamma_X x = \Gamma_X (K_\lambda \Gamma_Y y) = \Gamma_Y y$ , that is  $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{D}(\mathcal{A}_0)$ . Hence  $\mathcal{D}(\mathcal{B}_\lambda) \subset \mathcal{D}(\mathcal{A}_0)$ , and the proof is complete.  $\square$

The main result of this section is the following theorem.

**Theorem 3.2** Assume that the conditions (i)–(viii) are satisfied. Then  $\mathcal{A}_0$  is closable in  $\mathcal{X} = X \times Y$  if and only if, for some  $\lambda \in \rho(A_1)$ , the operator  $M_\lambda = D + C K_\lambda \Gamma_Y - C_\lambda B$  is closable as an operator in  $Y$ . In this case, if  $\overline{M}_\lambda$  denotes this closure, the closure  $\mathcal{A}$  of  $\mathcal{A}_0$  can be described as follows: for arbitrary  $\lambda \in \rho(A_1)$ ,

$$\begin{aligned} \mathcal{D}(\mathcal{A}) &= \left\{ \begin{pmatrix} x + \overline{K}_\lambda \overline{\Gamma}_Y^\circ y - \overline{(A_1 - \lambda I)^{-1} B} y \\ y \end{pmatrix} \mid x \in \mathcal{D}(A_1), y \in \mathcal{D}(\overline{M}_\lambda) \right\}, \\ \mathcal{A} \begin{pmatrix} x + \overline{K}_\lambda \overline{\Gamma}_Y^\circ y - \overline{(A_1 - \lambda I)^{-1} B} y \\ y \end{pmatrix} &= \begin{pmatrix} A_1 x + \lambda \overline{K}_\lambda \overline{\Gamma}_Y^\circ y - \lambda \overline{(A_1 - \lambda I)^{-1} B} y \\ C x + \overline{M}_\lambda y \end{pmatrix}. \end{aligned} \quad (3.3)$$

**Proof.** Under the assumptions of the theorem the operators

$$\begin{pmatrix} I & 0 \\ C_\lambda & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I & -\overline{K}_\lambda \overline{\Gamma}_Y^\circ + \overline{(A_1 - \lambda I)^{-1} B} \\ 0 & I \end{pmatrix}$$

are bounded and boundedly invertible as mappings from  $\mathcal{X}$  onto  $\mathcal{X}$ . Recalling the Frobenius–Schur factorization of  $\mathcal{A}_0 - \lambda \mathcal{I}$  in Lemma 3.1, we deduce that  $\mathcal{A}_0$  is closable in  $\mathcal{X}$  if and only if  $M_\lambda$  is closable as a mapping in  $Y$ . Moreover, if  $M_\lambda$  is closable and  $\overline{M}_\lambda$  denotes its closure, then for the closure  $\mathcal{A}$  of  $\mathcal{A}_0$  we obtain the relation

$$\mathcal{A} - \lambda \mathcal{I} = \begin{pmatrix} I & 0 \\ C_\lambda & I \end{pmatrix} \begin{pmatrix} A_1 - \lambda I & 0 \\ 0 & \overline{M}_\lambda - \lambda I \end{pmatrix} \begin{pmatrix} I & -\overline{K}_\lambda \overline{\Gamma}_Y^\circ + \overline{(A_1 - \lambda I)^{-1} B} \\ 0 & I \end{pmatrix}. \quad (3.4)$$

Spelling out the domain and the action of  $\mathcal{A}$  componentwise, the relation (3.3) follows. Clearly,  $\overline{\mathcal{A}_0 - \lambda \mathcal{I}} = \overline{\mathcal{A}_0} - \lambda \mathcal{I}$ , which implies that the definition of  $\mathcal{A}$  is independent of  $\lambda$ .  $\square$

The fact that the operator  $\mathcal{A}$  in Theorem 3.2 is well defined in the sense that both  $\mathcal{D}(\mathcal{A})$  and  $\mathcal{A} \mathbf{u}$  for  $\mathbf{u} \in \mathcal{D}(\mathcal{A})$  are independent of  $\lambda \in \rho(A_1)$  can also be seen explicitly. For  $\mathcal{D}(\mathcal{A})$  this follows if we observe that

- (a)  $\mathcal{D}(\overline{M}_\lambda)$  is independent of  $\lambda \in \rho(A_1)$  according to (2.12);
- (b)  $\overline{K}_{\lambda_1} z - \overline{K}_{\lambda_2} z \in \mathcal{D}(A_1)$  for  $z \in \overline{Z}_1$  and  $\lambda_1, \lambda_2 \in \rho(A_1)$  by Lemma 2.4;
- (c)  $\overline{(A_1 - \lambda_1 I)^{-1} B} y - \overline{(A_1 - \lambda_2 I)^{-1} B} y \in \mathcal{D}(A_1)$  for  $y \in Y$  and  $\lambda_1, \lambda_2 \in \rho(A_1)$  by (2.9).

Concerning  $\mathcal{A} \mathbf{u}$ , suppose that

$$x_1 + \overline{K}_{\lambda_1} \overline{\Gamma}_Y^\circ y - \overline{(A_1 - \lambda_1 I)^{-1} B} y = x_2 + \overline{K}_{\lambda_2} \overline{\Gamma}_Y^\circ y - \overline{(A_1 - \lambda_2 I)^{-1} B} y$$

are two representations for the first component of a vector  $\mathbf{u} \in \mathcal{D}(\mathcal{A})$ . Then from Lemma 2.4 and (2.9) it follows that

$$(x_1 - x_2) + (\lambda_1 - \lambda_2)(A_1 - \lambda_1 I)^{-1} [\overline{K}_{\lambda_2} \overline{\Gamma}_Y^\circ - \overline{(A_1 - \lambda_2 I)^{-1} B}] y = 0.$$

Applying the operator  $C$  and using (2.12), we get

$$C(x_1 - x_2) + \overline{M}_{\lambda_1}y - \overline{M}_{\lambda_2}y = 0.$$

In a similar way one shows that

$$A_1x_1 + \lambda_1\overline{K}_{\lambda_1}\overline{\Gamma}_Y^\circ y - \lambda_1\overline{(A_1 - \lambda_1 I)^{-1}B}y = A_1x_2 + \lambda_2\overline{K}_{\lambda_2}\overline{\Gamma}_Y^\circ y - \lambda_2\overline{(A_1 - \lambda_2 I)^{-1}B}y,$$

and hence  $\mathcal{A}\mathbf{u}$  is independent of the representation of  $\mathbf{u}$ .

#### 4 The essential spectrum of $\mathcal{A}$

Recall that an operator  $S$  acting in a Banach space  $X$  is called *Fredholm* if  $\mathcal{N}(S)$  has finite dimension and  $\mathcal{R}(S)$  is closed and has finite codimension in  $X$ , in which case the number

$$\text{ind } S := \dim \mathcal{N}(S) - \text{codim } \mathcal{R}(S)$$

is called the *index* of the Fredholm operator  $S$ . A point  $\lambda \in \mathbb{C}$  belongs to the essential spectrum  $\sigma_{\text{ess}}(T)$  of a closed operator  $T$  if and only if  $T - \lambda I$  is not a Fredholm operator in  $X$ .

**Lemma 4.1** *Assume that for some (and hence for all)  $\mu \in \rho(A_1)$  the operator  $M_\mu = D + CK_\mu\Gamma_Y - C_\mu B$  is closable and that the operator  $C$  is  $A_1$ -compact. Then the essential spectrum  $\sigma_{\text{ess}}(\overline{M}_\mu)$  of  $\overline{M}_\mu$  and (for  $\lambda \notin \sigma_{\text{ess}}(\overline{M}_\mu)$ ) the index  $\text{ind}(\overline{M}_\mu - \lambda I)$  are both independent of  $\mu \in \rho(A_1)$ .*

**Proof.** If  $\lambda, \lambda_1 \in \rho(A_1)$  then relation (2.12) holds. Since  $C$  is  $A_1$ -compact, the operator  $C_{\lambda_1}$  in (2.12) is compact. The claims now follow from the fact that the essential spectrum and the index of an operator do not change under a compact perturbation, see [17, Theorems IV.5.26 and IV.5.35].  $\square$

In the sequel, we denote by  $\tilde{\rho}(A_1)$  the union of  $\rho(A_1)$  and the discrete spectrum of  $A_1$ , i.e.,  $\tilde{\rho}(A_1)$  is the set of all points which are either regular points for  $A_1$  or isolated eigenvalues with a finite-dimensional Riesz projection (cf. [15, pp. 8–9] where such eigenvalues are called normal).

**Theorem 4.2** (1) *If the operator  $M_\mu$  is closable for some (and hence for all)  $\mu \in \rho(A_1)$  and the operator  $C$  is  $A_1$ -compact, then*

$$\sigma_{\text{ess}}(\mathcal{A}) \cap \tilde{\rho}(A_1) = \sigma_{\text{ess}}(\overline{M}_\mu) \cap \tilde{\rho}(A_1); \quad (4.1)$$

moreover, for all  $\mu \in \rho(A_1)$  and all  $\lambda \in \tilde{\rho}(A_1) \setminus \sigma_{\text{ess}}(\mathcal{A})$ ,

$$\text{ind}(\mathcal{A} - \lambda I) = \text{ind}(\overline{M}_\mu - \lambda I).$$

(2) *Assume in addition that, for some  $\lambda_0 \in \rho(A_1)$ , the operator  $\overline{K}_{\lambda_0}$  is compact as a mapping from  $\overline{Z}_1$  into  $X$  and the operator  $\overline{(A_1 - \lambda_0 I)^{-1}B}$  is compact as a mapping from  $Y$  into  $X$ . Then*

$$\sigma_{\text{ess}}(\mathcal{A}) = \sigma_{\text{ess}}(\overline{M}_\mu) \cup \sigma_{\text{ess}}(A_1)$$

and

$$\text{ind}(\mathcal{A} - \lambda I) = \text{ind}(\overline{M}_\mu - \lambda I) + \text{ind}(A_1 - \lambda I)$$

if  $\mu \in \rho(A_1)$  and  $\lambda \notin \sigma_{\text{ess}}(\mathcal{A})$ .

**Proof.** (1) Suppose first that  $\lambda \in \rho(A_1)$ . According to the Frobenius–Schur factorization (3.4),

$$\mathcal{A} - \lambda I = \mathcal{F}_\lambda \begin{pmatrix} A_1 - \lambda I & 0 \\ 0 & \overline{M}_\lambda - \lambda I \end{pmatrix} \mathcal{G}_\lambda,$$



where the external factors  $\mathcal{F}_\lambda$  and  $\mathcal{G}_\lambda$  are bounded and boundedly invertible in  $\mathcal{X}$ . Therefore  $\mathcal{A} - \lambda\mathcal{I}$  is Fredholm if and only if  $\overline{M}_\lambda - \lambda I$  has this property. Thus  $\lambda \notin \sigma_{\text{ess}}(\mathcal{A})$  if and only if  $\lambda \notin \sigma_{\text{ess}}(\overline{M}_\lambda) = \sigma_{\text{ess}}(\overline{M}_\mu)$ , in which case

$$\text{ind}(\mathcal{A} - \lambda\mathcal{I}) = \text{ind}(\overline{M}_\mu - \lambda I).$$

Now (4.1) follows (with  $\tilde{\rho}$  replaced by  $\rho$ ) since  $\lambda$  is an arbitrary point of  $\rho(A_1)$ .

To complete the proof of (1), we assume that  $\lambda$  belongs to the discrete spectrum of  $A_1$ . We shall show that again  $\lambda \in \sigma_{\text{ess}}(\mathcal{A})$  if and only if  $\lambda \in \sigma_{\text{ess}}(\overline{M}_\mu)$ . To this end we construct a finite rank perturbation  $\tilde{A}_1$  of  $A_1$ , so that  $\lambda$  is in the resolvent set  $\rho(\tilde{A}_1)$  and which is such that the perturbed operators  $\tilde{\mathcal{A}}$  and  $\tilde{M}_\lambda$  have the same essential spectra as the operators  $\mathcal{A}$  and  $\overline{M}_\lambda$ . Then the statement will follow from the first part of the proof.

The details are as follows. Since  $\lambda$  belongs to the discrete spectrum of  $A_1$  there exists an  $\varepsilon > 0$  such that the disk  $\{\zeta \in \mathbb{C} \mid |\zeta - \lambda| \leq 2\varepsilon\}$  does not contain points of  $\sigma(A_1)$  different from  $\lambda$ , and the Riesz projection  $P$  of  $A_1$  corresponding to  $\lambda$  is of finite rank. Consider the operator  $\tilde{A}_1 := A_1 + \varepsilon P$ . Then

$$\{\mu \in \mathbb{C} \mid 0 < |\mu - \lambda| < \varepsilon\} \subset \rho(A_1) \cap \rho(\tilde{A}_1).$$

Until further notice we fix  $\mu \in \rho(A_1) \cap \rho(\tilde{A}_1)$ .

The operator  $\tilde{\mathcal{A}}$  is defined as  $\mathcal{A}$  but with  $A$  replaced by  $\tilde{A} := A + \varepsilon P$ , so

$$\tilde{\mathcal{A}}_0 = \begin{pmatrix} \tilde{A} & B \\ C & D \end{pmatrix} = \mathcal{A}_0 + \varepsilon \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix},$$

and for the closure we obtain

$$\tilde{\mathcal{A}} = \mathcal{A} + \varepsilon \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix},$$

which is a finite rank perturbation of  $\mathcal{A}$ . Therefore  $\sigma_{\text{ess}}(\tilde{\mathcal{A}}) = \sigma_{\text{ess}}(\mathcal{A})$  and  $\text{ind}(\tilde{\mathcal{A}} - \lambda\mathcal{I}) = \text{ind}(\mathcal{A} - \lambda\mathcal{I})$  for  $\lambda \notin \sigma_{\text{ess}}(\mathcal{A})$ . We also denote by  $\tilde{K}_\mu$ ,  $\mu \in \rho(\tilde{A}_1)$ , the operator in (2.5) with  $A$  replaced by  $\tilde{A}$ . This means that  $\tilde{K}_\mu z = x$  is equivalent to  $\tilde{A}x = \mu x$  and  $\tilde{\Gamma}_X x = z$ . The operator  $\tilde{K}_\mu$  is well defined for  $\mu \in \rho(\tilde{A}_1)$ .

We claim that the difference  $\tilde{K}_\mu - K_\mu$  is of finite rank. To see this, take  $z \in Z_1$  and put  $\tilde{u} = \tilde{K}_\mu z$ ,  $u = K_\mu z$ . Then  $\tilde{u} - u$  satisfies the relations  $\tilde{\Gamma}_X(\tilde{u} - u) = 0$  and  $(A - \mu I)(\tilde{u} - u) = -\varepsilon P\tilde{u}$ . This implies that  $\tilde{u} - u \in \mathcal{D}(A_1)$  and  $\tilde{u} - u = -(A_1 - \mu I)^{-1}\varepsilon P\tilde{u}$ , so that  $\tilde{K}_\mu - K_\mu = -\varepsilon P(A_1 - \mu I)^{-1}\tilde{K}_\mu$ , and our claim is established.

It is easy to see that the closure of the difference

$$C_\mu B - C(\tilde{A}_1 - \mu I)^{-1}B$$

is also of finite rank. By assumption, the operator  $M_\mu$  is closable in  $Y$ , so its perturbation  $\tilde{M}_\mu := D + C\tilde{K}_\mu\Gamma_X - C(\tilde{A}_1 - \mu I)^{-1}B$  is closable in  $Y$  as well; we denote this closure by  $\widehat{M}_\mu$ . Since  $\widehat{M}_\mu - \overline{M}_\mu$  is of finite rank, the essential spectra of  $\widehat{M}_\mu$  and  $\overline{M}_\mu$  coincide. Now we observe that  $C$  is also  $\tilde{A}_1$ -compact, so Lemma 4.1 implies that  $\widehat{M}_\mu$  is closable for all  $\mu \in \rho(\tilde{A}_1)$  and that the essential spectra of the closures  $\widehat{M}_\mu$  are independent of  $\mu \in \rho(\tilde{A}_1)$ , and thus coincide with the set  $\sigma_{\text{ess}}(\overline{M}_\mu)$ . Now applying the first part of this proof but for  $\lambda \in \rho(\tilde{A}_1)$  instead, we see that  $\lambda \in \sigma_{\text{ess}}(\tilde{\mathcal{A}}) = \sigma_{\text{ess}}(\mathcal{A})$  if and only if  $\lambda$  belongs to  $\sigma_{\text{ess}}(\overline{M}_\mu)$ . If  $\lambda \in \rho(\tilde{A}_1) \setminus \sigma_{\text{ess}}(\overline{M}_\mu)$ , then we also find that

$$\text{ind}(\mathcal{A} - \lambda\mathcal{I}) = \text{ind}(\tilde{\mathcal{A}} - \lambda\mathcal{I}) = \text{ind}(\widehat{M}_\lambda - \lambda I) = \text{ind}(\overline{M}_\mu - \lambda I)$$

for any  $\mu \in \rho(A_1)$  as required, and the proof of (1) is complete.

(2) Following [25], we fix  $\lambda_0 \in \rho(A_1)$  and find that

$$\mathcal{A} - \lambda\mathcal{I} = \mathcal{A} - \lambda_0\mathcal{I} + (\lambda_0 - \lambda)\mathcal{I} =$$

$$\begin{aligned}
&= \mathcal{F}_{\lambda_0} \begin{pmatrix} A_1 - \lambda_0 I & 0 \\ 0 & \overline{M}_{\lambda_0} - \lambda_0 I \end{pmatrix} \mathcal{G}_{\lambda_0} + (\lambda_0 - \lambda) \mathcal{I} \\
&= \mathcal{F}_{\lambda_0} \begin{pmatrix} A_1 - \lambda I & 0 \\ 0 & \overline{M}_{\lambda_0} - \lambda I \end{pmatrix} \mathcal{G}_{\lambda_0} - (\lambda_0 - \lambda) (\mathcal{F}_{\lambda_0} \mathcal{G}_{\lambda_0} - \mathcal{I}).
\end{aligned}$$

Under the additional assumptions that both  $\overline{K}_{\lambda_0}$  and  $\overline{(A_1 - \lambda_0 I)^{-1} B}$  are compact, we observe that the operator  $\mathcal{F}_{\lambda_0} \mathcal{G}_{\lambda_0} - \mathcal{I}$  is compact. Therefore the operator  $\mathcal{A} - \lambda \mathcal{I}$  is Fredholm if and only if  $A_1 - \lambda I$  and  $\overline{M}_{\lambda_0} - \lambda I$  are Fredholm and in that case

$$\operatorname{ind}(\mathcal{A} - \lambda \mathcal{I}) = \operatorname{ind}(\overline{M}_{\lambda_0} - \lambda I) + \operatorname{ind}(A_1 - \lambda I)$$

as claimed.  $\square$

**Remark 4.3** The two compactness assumptions in (2) can be weakened so that they are taken relative to  $A_1$  and  $\overline{M}_{\lambda_0}$  respectively.

## 5 The operator $\mathcal{A}$ as an infinitesimal generator of a holomorphic semigroup

For this section we suppose besides the assumptions (i)–(viii) that for some  $\lambda_0 \in \rho(A_1)$  the operator  $M_{\lambda_0}$  is closable. The Frobenius–Schur factorization implies that the resolvent growth of  $\mathcal{A}$  is essentially determined by the growth of  $(A_1 - \lambda I)^{-1}$  and of  $(\overline{M}_{\lambda} - \lambda I)^{-1}$ . The following theorem gives some sufficient conditions for  $\mathcal{A}$  to be the generator of a holomorphic semigroup (see, e.g., [13, Sect. II.4.a] for details) in terms of  $A_1$  and  $\overline{M}_{\lambda}$ .

We recall first the following criterion (cf. [13, Theorem II.4.6]). An operator  $T$  in a Banach space generates a holomorphic semigroup of semiangle  $\theta \in (0, \frac{\pi}{2})$  if and only if there exists  $\omega \in \mathbb{R}$  such that the sector

$$S(\omega, \theta) := \left\{ \lambda \in \mathbb{C} \mid |\arg(\lambda - \omega)| < \frac{\pi}{2} + \theta \right\} \quad (5.1)$$

belongs to  $\rho(T)$  and for each  $\varepsilon \in (0, \theta)$  there is  $L_\varepsilon \geq 1$  such that the resolvent of  $T$  satisfies the inequality

$$\|(T - \lambda I)^{-1}\| \leq L_\varepsilon / |\lambda - \omega| \quad (5.2)$$

in  $S(\omega, \theta - \varepsilon)$ . We mention that it suffices to prove inequality (5.2) for all  $\lambda \in S(\omega, \theta)$  with sufficiently large  $|\lambda|$ . Also, shifting the eigenvalue parameter  $\lambda$  if necessary, we may assume that  $\omega > 0$ , and then inequality (5.2) may be replaced by the following one:

$$\sup_{\lambda \in S(\omega, \theta)} \|\lambda(T - \lambda I)^{-1}\| < \infty.$$

**Theorem 5.1** *In addition to the assumptions at the beginning of this section, suppose that  $A_1$  and  $\overline{M}_{\lambda_0}$  generate holomorphic semigroups in  $X$  and  $Y$ , respectively, and, for some  $\omega > 0$ , let  $S(\omega, \theta)$  be a sector (5.1) corresponding to the operator  $A_1$ . If condition (iv) is strengthened to*

$$\inf_{\lambda \in S(\omega, \theta)} \|C_\lambda\| = 0, \quad (5.3)$$

*then the operator  $\mathcal{A}$  generates a holomorphic semigroup in  $\mathcal{X}$ .*

**Proof.** We shall verify that the conditions in the criterion quoted above are satisfied for the operator  $\mathcal{A}$ .

It follows from (3.4) that for  $\lambda \in \rho(A_1)$  the operator  $\mathcal{A} - \lambda \mathcal{I}$  is boundedly invertible if and only if  $\overline{M}_{\lambda} - \lambda I$  has this property, and that in this case

$$(\mathcal{A} - \lambda \mathcal{I})^{-1} = \begin{pmatrix} I & \overline{K}_\lambda \overline{\Gamma}_Y - \overline{(A_1 - \lambda I)^{-1} B} \\ 0 & I \end{pmatrix} \begin{pmatrix} (A_1 - \lambda I)^{-1} & 0 \\ 0 & (\overline{M}_\lambda - \lambda I)^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -C_\lambda & I \end{pmatrix}. \quad (5.4)$$

Since the operators  $A_1$  and  $\overline{M}_{\lambda_0}$  are supposed to generate holomorphic semigroups, there exist constants  $L_1 > 0$ ,  $\theta_1 \in (0, \theta)$ , and  $\omega_1 > 0$  such that the sector  $S(\omega_1, \theta_1)$  belongs to the resolvent set of both  $A_1$  and  $\overline{M}_{\lambda_0}$  and that, for all  $\lambda \in S(\omega_1, \theta_1)$ ,

$$\|(A_1 - \lambda I)^{-1}\| \leq L_1/|\lambda|, \quad \|(\overline{M}_{\lambda_0} - \lambda I)^{-1}\| \leq L_1/|\lambda|. \quad (5.5)$$

Without loss of generality we may assume that  $\omega = \omega_1$ .

It follows from (5.5) that

$$\|A_1(A_1 - \lambda I)^{-1}\| = \|I + \lambda(A_1 - \lambda I)^{-1}\| \leq L_1 + 1 \quad (5.6)$$

for  $\lambda \in S(\omega, \theta_1)$ . With  $\lambda_1 \in \rho(A_1)$  fixed, we have

$$C_\lambda = C_{\lambda_1} [A_1(A_1 - \lambda I)^{-1} - \lambda_1(A_1 - \lambda I)^{-1}]$$

so

$$\|C_\lambda\| \leq \|C_{\lambda_1}\| (2L_1 + 1) \quad (5.7)$$

provided  $|\lambda| > |\lambda_1|$ , by (5.5) and (5.6). Since  $\|C_{\lambda_1}\|$  can be made arbitrarily small in view of condition (5.3), we see that

$$\lim_{|\lambda| \rightarrow \infty, \lambda \in S(\omega, \theta_1)} \|C_\lambda\| = 0. \quad (5.8)$$

Using the first inequality in (5.5), Lemma 2.4 and (2.9), we also conclude that the operators  $\overline{(A_1 - \lambda I)^{-1}B}$  and  $\overline{K_\lambda \Gamma_Y^0}$  are uniformly bounded in  $\lambda \in S(\omega, \theta_1)$ . Therefore the factors

$$\begin{pmatrix} I & \overline{K_\lambda \Gamma_Y} - \overline{(A_1 - \lambda I)^{-1}B} \\ 0 & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I & 0 \\ -C_\lambda & I \end{pmatrix}$$

in (5.4) are uniformly bounded in  $\lambda \in S(\omega, \theta_1)$ .

Next, by virtue of (2.12) we have

$$\begin{aligned} \overline{M}_\lambda - \lambda I &= (\overline{M}_{\lambda_0} - \lambda I) \left( I + (\overline{M}_{\lambda_0} - \lambda I)^{-1} (\overline{M}_\lambda - \overline{M}_{\lambda_0}) \right) \\ &= (\overline{M}_{\lambda_0} - \lambda I) \left( I + (\overline{M}_{\lambda_0} - \lambda I)^{-1} (\lambda - \lambda_0) C_\lambda (\overline{K}_{\lambda_0} \overline{\Gamma}_Y^0 - \overline{(A_1 - \lambda_0 I)^{-1}B}) \right). \end{aligned}$$

The second estimate in (5.5) implies that

$$\sup_{\lambda \in S(\omega, \theta_1)} \|(\overline{M}_{\lambda_0} - \lambda I)^{-1} (\lambda - \lambda_0)\| < \infty,$$

so by (5.8) there exists an  $r > 0$  such that

$$\|(\overline{M}_{\lambda_0} - \lambda I)^{-1} (\lambda - \lambda_0) C_\lambda (\overline{K}_{\lambda_0} \overline{\Gamma}_Y^0 - \overline{(A_1 - \lambda_0 I)^{-1}B})\| < 1/2 \quad (5.9)$$

for all  $\lambda \in S(\omega, \theta_1)$  with  $|\lambda| > r$ . Thus, for these  $\lambda$ , the operator

$$\left( I + (\overline{M}_{\lambda_0} - \lambda I)^{-1} (\lambda - \lambda_0) C_\lambda (\overline{K}_{\lambda_0} \overline{\Gamma}_Y^0 - \overline{(A_1 - \lambda_0 I)^{-1}B}) \right)$$

is invertible and the norm of its inverse is not greater than 2. We arrive at the conclusion that

$$\|\lambda(\overline{M}_\lambda - \lambda I)^{-1}\| \leq 2 \|\lambda(\overline{M}_{\lambda_0} - \lambda I)^{-1}\| \leq 2L_1$$

for all  $\lambda \in S(\omega, \theta_1)$  with  $|\lambda| > r$ .

Combining these estimates, we see that  $\|\lambda(\mathcal{A} - \lambda \mathcal{I})^{-1}\|$  is bounded uniformly for all  $\lambda \in S(\omega + r, \theta_1)$ , and therefore the operator  $\mathcal{A}$  generates a holomorphic semigroup in  $\mathcal{X}$ .  $\square$

**Remark 5.2** It is readily seen that assumption (5.3) can be replaced by the weaker condition that the infimum is less than a constant  $c$  chosen small enough for the norm in (5.9) to be less than  $d$ , say, where  $d < 1$ .

**Proposition 5.3** Assume that  $A_1$  generates a holomorphic semigroup and  $S(\omega, \theta)$  is the corresponding sector. Then (5.3) holds if the relative  $A_1$ -bound of  $C$  is zero. This is the case if  $C$  is  $A_1$ -compact and either  $X$  is reflexive or  $C$  (as originally defined, i.e., from  $X$  to  $Y$ ) is closable.

*Proof.* The relative  $A_1$ -boundedness condition implies that for every  $b > 0$  there is  $a > 0$  such that

$$\|Cx\| \leq a\|x\| + b\|A_1x\| \quad (5.10)$$

for all  $x \in \mathcal{D}(A_1)$ .

Choosing  $x = (A_1 - \lambda I)^{-1}u$  in (5.10) with  $\lambda$  in the sector  $S(\omega, \theta)$  (as in the proof of Theorem 5.1) and using (5.6), we get

$$\|C_\lambda u\| \leq a\|(A_1 - \lambda I)^{-1}u\| + b\|A_1(A_1 - \lambda I)^{-1}u\| \leq \left\{ \frac{aL_1}{|\lambda|} + b(L_1 + 1) \right\} \|u\|.$$

The right side can be made arbitrarily small if we choose  $b$  sufficiently small and then  $|\lambda|$  large. This yields the first claim.

For the remainder, we observe that if  $C$  is  $A_1$ -compact and  $X$  is reflexive or  $C$  is closable, then the relative  $A_1$ -bound of  $C$  is zero—see, e.g., [7, Theorem 2] or [13, Lemma III.2.16].  $\square$

## 6 A $\lambda$ -rational Sturm–Liouville problem

Let  $p, u \in L_1(0, 1)$  be real,  $q \in L_2(0, 1)$  be real,  $\sigma$  be a bounded nonnegative measure on  $\mathbb{R}$ , and  $\alpha_j, \beta_j \in \mathbb{R}$ ,  $\alpha_j^2 + \beta_j^2 \neq 0$ ,  $j = 0, 1$ . We consider the spectral problem

$$-f'' + pf + \frac{qf}{u - \lambda} - \lambda f = 0 \quad \text{on } [0, 1], \quad (6.1)$$

$$b_1(f) + N(\lambda)b_0(f) = 0, \quad f(1) = 0, \quad (6.2)$$

where  $N(\lambda)$  is the Nevanlinna function

$$N(\lambda) := \int_{\mathbb{R}} \frac{d\sigma(t)}{t - \lambda}$$

and

$$b_j(f) := \alpha_j f'(0) + \beta_j f(0), \quad j = 0, 1.$$

These equations make sense at least for  $\lambda \in \mathbb{C} \setminus \mathbb{S}$ , where  $\mathbb{S} := \overline{u([0, 1])} \cup \text{supp } \sigma$ . If  $q = 0$  then we can take  $\mathbb{S} = \text{supp } \sigma$  and also the following considerations can be simplified. A point  $\lambda \in \mathbb{C} \setminus \mathbb{S}$  is called an *eigenvalue* of the problem (6.1), (6.2) if there exists a nonzero function  $f$  such that  $f'$  is absolutely continuous and these equations are satisfied.

We associate the following operator matrix with the problem (6.1), (6.2). Set  $X = L_2(0, 1)$ ,  $Y = L_2(0, 1) \times \mathcal{H}$ , where  $\mathcal{H}$  is the Hilbert space  $L_2(\sigma; \mathbb{R})$  of all functions  $g$  on  $\mathbb{R}$  with

$$\|g\|^2 = \int_{\mathbb{R}} |g(t)|^2 d\sigma(t) < \infty,$$

and  $Z = \mathbb{C}$ . Define operators  $A, B, C, D, \Gamma_X, \Gamma_Y$  as follows:

$$Af = -f'' + pf \quad \text{on } \mathcal{D}(A) = \{f \in W_2^2[0, 1] \mid f(1) = 0\}, \quad (6.3)$$

$$B \begin{pmatrix} h \\ g \end{pmatrix} = qh, \quad Cf = \begin{pmatrix} -f \\ b_0(f) \end{pmatrix}, \quad D \begin{pmatrix} h \\ g \end{pmatrix} = \begin{pmatrix} Uh \\ Tg \end{pmatrix},$$

$$\Gamma_X f = b_1(f), \quad \Gamma_Y \begin{pmatrix} h \\ g \end{pmatrix} = \int_{\mathbb{R}} g(t) d\sigma(t),$$

where  $(Uh)(t) = u(t)h(t)$  a.e. in  $[0, 1]$  and  $(Tg)(t) = tg(t)$   $\sigma$ -a.e. on  $\mathbb{R}$ . The operator  $D$  is defined on all vectors  $\begin{pmatrix} h \\ g \end{pmatrix}$ ,  $h \in L_2(0, 1)$ ,  $g \in \mathcal{H}$ , such that  $Uh \in L_2(0, 1)$  and  $Tg \in \mathcal{H}$ . It is well-known that the operator  $A$  is densely defined and closed (so (i) holds), and that the functionals  $b_0$  and  $b_1$  are defined and continuous on  $X_A$ , which implies (ii). The operator  $A_1$  is the (self-adjoint) restriction of  $A$  by the boundary condition  $b_1(f) = 0$ , whence (iii) is satisfied, and since the embedding of  $X_A$  into  $L_2(0, 1)$  is continuous and  $b_0$  is continuous on  $X_A$ ,  $C$  is continuous as an operator from  $X_A$  into  $Y$  and thus (iv) holds. The operator  $K_\lambda$  is defined at least for nonreal  $\lambda$  and, for  $z \in \mathbb{C}$ ,  $K_\lambda z$  is the solution of the boundary value problem

$$-f'' + pf = \lambda f, \quad b_1(f) = z, \quad f(1) = 0,$$

which depends continuously on  $z$  with respect to the norm of  $L_2(0, 1)$ , so (v) holds. Evidently,  $D$  is self-adjoint in  $Y$ , and hence (vi) holds. The estimate  $\left| \int_{\mathbb{R}} g(t) d\sigma(t) \right|^2 \leq \int_{\mathbb{R}} |g(t)|^2 d\sigma(t) \int_{\mathbb{R}} d\sigma(t)$  implies that condition (vii) is satisfied. Finally, since the operator  $(A_1 - \lambda)^{-1}$  acts boundedly from  $L_1(0, 1)$  into  $L_2(0, 1)$ , the assumption  $q \in L_2(0, 1)$  implies the boundedness of the operator  $(A_1 - \lambda)^{-1}B$ , and hence (viii) holds for nonreal  $\lambda$ .

If  $f \in \mathcal{D}(A)$  and  $\begin{pmatrix} h \\ g \end{pmatrix} \in \mathcal{D}(D) \cap \mathcal{D}(B)$  then

$$\mathcal{A}_0 \begin{pmatrix} f \\ h \\ g \end{pmatrix} = \begin{pmatrix} Af + B \begin{pmatrix} h \\ g \end{pmatrix} \\ Cf + D \begin{pmatrix} h \\ g \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -f'' + pf + qh \\ \begin{pmatrix} -f \\ b_0(f) \end{pmatrix} + \begin{pmatrix} Uh \\ Tg \end{pmatrix} \end{pmatrix}, \quad f(1) = 0,$$

and the equation  $\mathcal{A}_0 \begin{pmatrix} f \\ h \\ g \end{pmatrix} = \lambda \begin{pmatrix} f \\ h \\ g \end{pmatrix}$  becomes

$$-f'' + pf + qh = \lambda f, \quad f(1) = 0, \quad -f + uh = \lambda h, \quad b_0(f) + Tg = \lambda g. \quad (6.4)$$

Finally,

$$\Gamma_X f = \Gamma_Y \begin{pmatrix} h \\ g \end{pmatrix}$$

yields

$$b_1(f) = \int_{\mathbb{R}} g(t) d\sigma(t). \quad (6.5)$$

Solving the last two equations in (6.4) for  $h$  and  $g$  we get

$$h = \frac{f}{u - \lambda}, \quad g(t) = -\frac{b_0(f)}{t - \lambda},$$

and the first two equations in (6.4) and (6.5) give

$$-f'' + pf + \frac{qf}{u - \lambda} = \lambda f, \quad f(1) = 0, \quad b_1(f) + N(\lambda)b_0(f) = 0,$$

and these are the relations (6.1) and (6.2).

The operator  $\mathcal{A}$  can now be defined as in Section 3. Since  $A_1$  has compact resolvent,  $C$  is  $A_1$ -compact and  $\overline{K}_\lambda$ ,  $(A_1 - \lambda I)^{-1}B$  are also compact operators. Thus the assumptions of Theorem 4.2, (2) are satisfied. Since the spectrum of  $A_1$  is discrete, the essential spectrum of  $\mathcal{A}$  coincides with the essential spectrum of  $\overline{M}_\mu$  (and therefore of  $D$ ). This in turn is the union of the essential spectra of the operators  $U$  and  $T$ , i.e., the union of the essential range of the function  $u$  and the support of the non-atomic part of the measure  $\sigma$ .

The above considerations in this section imply that in the set  $\mathbb{C} \setminus \mathbb{S}$  the eigenvalues of the operator  $\mathcal{A}$  coincide with the eigenvalues of the problem (6.1), (6.2). The operator  $\mathcal{A}$  can be considered as a *linearization* of the  $\lambda$ -rational eigenvalue problem (6.1), (6.2). In fact, if we introduce for  $\lambda \in \mathbb{C} \setminus \mathbb{S}$  in  $L_2(0, 1)$  the operators  $T(\lambda)$  defined by

$$T(\lambda)f := -f'' + pf + \frac{qf}{u - \lambda} \quad \text{on } [0, 1],$$

with domain given by (6.2), then the relation

$$(T(\lambda) - \lambda I)^{-1} = P_X(\mathcal{A} - \lambda \mathcal{I})^{-1}|_X$$

holds for  $\lambda \in \rho(\mathcal{A}) \cap (\mathbb{C} \setminus \mathbb{S})$ , where  $P_X$  denotes the orthogonal projection in  $\mathcal{X}$  onto  $X$ . It is also natural to say that the points of  $\mathbb{S}$  are eigenvalues or belong to the continuous spectrum of the problem (6.1), (6.2) if they are eigenvalues or belong to the continuous spectrum of the operator  $\mathcal{A}$ . Such embedded eigenvalues of (6.1), (6.2) can also be characterized directly without using the linearization  $\mathcal{A}$  for the case  $\sigma = 0$ , cf. [18].

## 7 Elliptic problems with $\lambda$ -dependent boundary conditions

### 7.1 The problem

Let  $\Omega$  be an open bounded domain in  $\mathbb{R}^n$  with closure  $\overline{\Omega}$  and boundary  $\partial\Omega$  of class  $C^\infty$ , see [20, Section 1.7]. We refer the reader to [20, Section 2.1] for the following notions from the theory of elliptic operators.

Suppose that the operator

$$A(x, \partial) := - \sum_{j,k=1}^n a_{jk}(x) \partial_j \partial_k + \sum_{j=1}^n a_j(x) \partial_j + a_0(x)$$

with  $a_{jk} \in C(\overline{\Omega})$ ,  $a_0, a_j \in L^\infty(\overline{\Omega})$ ,  $j, k = 1, \dots, n$  is properly elliptic in  $\overline{\Omega}$ . This means that, for any  $x \in \overline{\Omega}$  and any linearly independent vectors  $\xi, \eta \in \mathbb{R}^n$ , the equation

$$A_0(x, \xi + t\eta) = 0 \tag{7.1}$$

has exactly one root  $t$  with positive imaginary part. Here

$$A_0(x, \xi) := \sum_{j,k=1}^n a_{jk}(x) \xi_j \xi_k \tag{7.2}$$

is the principal symbol of the operator  $A(x, \partial)$ .

The trace operator of the restriction of smooth functions in  $\Omega$  to the boundary  $\partial\Omega$  is denoted by  $\gamma$ . Let

$$\mathbf{B}_j = \sum_{k=1}^n \gamma b_{jk}(x) \partial_k + \gamma b_{j0}(x), \quad j = 0, 1,$$

be two boundary operators with  $b_{jk} \in C^1(\partial\Omega)$  and  $b_{j0} \in C^2(\partial\Omega)$  such that  $\mathbf{B}_1$  is normal on  $\partial\Omega$  and covers  $A = A(x, \partial)$ . That is, the problem  $\{A, \mathbf{B}_1\}$  is supposed to be regular elliptic in  $\Omega$ .

Let  $m(x, y)$  be a bounded measurable function of  $x \in \Omega$  and  $y \in \partial\Omega$ . We introduce an operator  $B$ , acting from  $L_2(\partial\Omega)$  into  $L_2(\Omega)$ , via

$$(Bg)(x) := \int_{\partial\Omega} m(x, y) g(y) dy, \quad x \in \Omega.$$

Finally, let  $D$  be a properly elliptic operator on  $\partial\Omega$  of second order with smooth coefficients. This means that in the local coordinate system  $\{y_1, y_2, \dots, y_{n-1}\}$  of the point  $y \in \partial\Omega$  the operator  $D$  is represented by

$$D(y) := - \sum_{j,k=1}^{n-1} d_{jk}(y) \partial_j \partial_k + \sum_{j=1}^{n-1} d_j(y) \partial_j + d_0(y),$$

where  $\partial_j := \frac{\partial}{\partial y_j}$  and  $d_{jk} \in C(\partial\Omega)$ ,  $d_0, d_j \in L_\infty(\partial\Omega)$ ,  $j, k = 1, \dots, n-1$ , and that a root condition analogous to (7.1) holds for  $D$ . A typical example of such a  $D$  is  $-\Delta_{\partial\Omega}$ , the negative Laplace–Beltrami operator on  $\partial\Omega$ .

We consider the following boundary value spectral problem:

$$Au + B\gamma u = \lambda u \quad \text{in } \Omega, \quad (7.3)$$

$$\mathbf{B}_0 u + D\gamma u = \lambda \mathbf{B}_1 u \quad \text{on } \partial\Omega. \quad (7.4)$$

The corresponding dynamical problem describes the motion of a Markovian particle that moves in  $\Omega$  according to a diffusion law and possibly jumps at random times from  $x \in \Omega$  into a set  $\Gamma_0 \subset \partial\Omega$  (if  $\int_{\Gamma_0} m(x, y) dy > 0$ ). After reaching the boundary (by jump or diffusion), the particle can be reflected into  $\Omega$ , it can be absorbed in  $\partial\Omega$ , or move in  $\partial\Omega$ ; this behaviour on the boundary is governed by the terms in the boundary condition—see [26].

It is easily seen that problem (7.3)–(7.4) can be represented in the form  $\mathcal{A}_0 \mathbf{u} = \lambda \mathbf{u}$  for the operator

$$\mathcal{A}_0 := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{in the Hilbert space } \mathcal{X} := L_2(\Omega) \oplus L_2(\partial\Omega)$$

with  $C = \mathbf{B}_0$ , and

$$\mathcal{D}(\mathcal{A}_0) := \left\{ \begin{pmatrix} u \\ g \end{pmatrix} \mid u \in H^2(\Omega), g \in H^2(\partial\Omega), \mathbf{B}_1 u = g \right\}.$$

The operator  $\mathcal{A}_0$  takes the form of the previous sections upon the following identification for the spaces and operators involved:  $X = L_2(\Omega)$ ,  $Y = L_2(\partial\Omega)$ ,  $Z = H^{-3/2}(\partial\Omega)$ ,  $A$ ,  $B$ , and  $D$  as stated above with  $\mathcal{D}(A) = H^2(\Omega)$  and  $\mathcal{D}(D) = H^2(\partial\Omega)$ ,  $C = \mathbf{B}_0$ ,  $\Gamma_X = \mathbf{B}_1$ , and  $\Gamma_Y$  being the natural embedding of  $L_2(\partial\Omega)$  into  $H^{-3/2}(\partial\Omega)$ .

## 7.2 Verification of assumptions (i)–(viii)

It follows from the general theory of elliptic operators that the operator  $A$  with domain  $H^2(\Omega)$  is closable in  $X = L_2(\Omega)$  and that its closure  $\overline{A}$  has domain

$$X_A = \{u \in X \mid Au \in X\},$$

where, as usual, the same notation  $A$  is used for the operator in the distributional sense, see [16]. For any  $u \in X_A$ ,  $\Gamma_X u$  exists as an element of  $Z = H^{-3/2}(\partial\Omega)$ , and the mapping  $\Gamma_X : X_A \rightarrow Z$  is bounded, see [16]. The same is of course true for the operator  $C$ ; as a result,  $C$  is closable as a mapping from  $X_A$  into  $Y = L_2(\partial\Omega)$ . These arguments establish properties (i), (ii), and (iv). Since  $D$  is a uniformly elliptic operator on a compact manifold without boundary,  $D$  is closed and has discrete spectrum, and hence (vi) is satisfied (see [27, Section 5.1] for the case when  $D$  is the Laplacian on  $\partial\Omega$  – the general case then follows from standard techniques). Also (vii) and (viii) are trivially satisfied here as  $\Gamma_Y$  is the embedding of  $Y$  into  $Z$  noted above and  $B$  is a bounded operator.

It remains to verify (iii) and (v). The first condition is a consequence of the following proposition.

**Proposition 7.1** (See [1, Theorem 2.1]) *Under the assumptions of Subsection 7.1, the operator  $A_1$  in  $L_2(\Omega)$ , defined on*

$$\mathcal{D}(A_1) := \{u \in H^2(\Omega) \mid B_1 u = 0\}$$

*by  $A_1 u := Au$ , is closed and has discrete spectrum.*

Suppose now that  $\lambda$  belongs to the resolvent set of  $A_1$ . It is known (see, e.g., [20, Ch. 2.7.3]) that for any  $g \in H^{1/2}(\partial\Omega)$  the problem

$$Au - \lambda u = 0, \quad \Gamma_X u = g$$

has a unique solution  $u$  which belongs to  $H^2(\Omega)$ . In particular, the operator  $K_\lambda$  mapping  $g \in H^{1/2}(\partial\Omega)$  to this solution  $u \in H^2(\Omega)$  is well defined. Thus we can identify the subspace  $Z_1$  of  $Z$  with  $H^{1/2}(\partial\Omega)$ . To verify (v) we show that  $K_\lambda$  extends to a bounded mapping  $\overline{K}_\lambda$  from  $Z$  into  $X$ . In fact, an even stronger result holds.

**Proposition 7.2** (See [20, Section 2.7.3]) *With the assumptions of Subsection 7.1, let  $\lambda \in \rho(A_1)$  and  $\ell \geq 0$ . Then for any  $g \in H^{\ell-3/2}(\partial\Omega)$  the problem*

$$Aw - \lambda w = 0, \quad B_1 w = g, \quad (7.5)$$

*has a unique solution  $w$ . Moreover,  $w$  belongs to  $H^\ell(\Omega)$  and the operator  $\overline{K}_\lambda : H^{\ell-3/2}(\partial\Omega) \rightarrow H^\ell(\Omega)$ , defined by  $\overline{K}_\lambda g = w$ , is bounded.*

For  $\ell = 0$  the above proposition yields the desired boundedness of  $\overline{K}_\lambda$ . Choosing  $\ell = 3/2$  and recalling that the embedding of  $H^{3/2}(\Omega)$  into  $X = L_2(\Omega)$  is compact, we arrive at the following corollary.

**Corollary 7.3**  $\overline{K}_\lambda$  is compact as an operator from  $Y = L_2(\partial\Omega)$  into  $X = L_2(\Omega)$ .

The above properties of  $\overline{K}_\lambda$  imply that for any  $y \in Y$  we have  $\overline{K}_\lambda \Gamma_Y y = \overline{K}_\lambda y \in H^{3/2}(\Omega)$ . In particular, we can write  $\overline{K}_\lambda$  instead of  $\overline{K}_\lambda \Gamma_Y$ .

### 7.3 The closure of the operator $\mathcal{A}_0$

In order to describe the closure  $\mathcal{A}$  of the operator  $\mathcal{A}_0$ , we start with the following lemma.

**Lemma 7.4** For any  $\lambda \in \rho(A_1)$  the operator  $C\overline{K}_\lambda$  is bounded in  $L_2(\partial\Omega)$ .

*Proof.* Since by assumption  $\mathbf{B}_1$  is normal on  $\partial\Omega$ , the vector field

$$\mathbf{b}_1(x) := (b_{11}(x) \dots b_{1n}(x))$$

is never tangential on  $\partial\Omega$ . Therefore there exist a continuous function  $\rho(x)$  and a vector field  $\mathbf{b}_t = (b_1 \dots b_n)$  which is tangential to  $\partial\Omega$  for  $x \in \partial\Omega$  such that

$$(b_{01}(x) \dots b_{0n}(x)) = \rho(x)\mathbf{b}_1(x) + \mathbf{b}_t(x).$$

Now we have

$$Cu = \rho(x)\Gamma_X u + \mathbf{B}_t u + b_0(x)\gamma u,$$

where

$$\mathbf{B}_t = \sum_{k=1}^n \gamma b_k(x) \partial_k, \quad b_0(x) = b_{00}(x) - \rho(x)b_{10}(x).$$

Since  $\mathbf{b}_t$  is tangential to  $\partial\Omega$ , the operator  $\mathbf{B}_t$  acts continuously from  $H^1(\partial\Omega)$  into  $L_2(\partial\Omega)$ . Next,  $\gamma\overline{K}_\lambda$  is continuous from  $L_2(\partial\Omega)$  into  $H^1(\partial\Omega)$  and hence  $\mathbf{B}_t\overline{K}_\lambda = \mathbf{B}_t\gamma\overline{K}_\lambda : L_2(\partial\Omega) \rightarrow L_2(\partial\Omega)$  is continuous. Finally,

$$C\overline{K}_\lambda = \rho\mathbf{B}_1\overline{K}_\lambda + \mathbf{B}_t\overline{K}_\lambda + b_0\gamma\overline{K}_\lambda = \rho I + \mathbf{B}_t\overline{K}_\lambda + b_0\gamma\overline{K}_\lambda. \quad (7.6)$$

□

**Corollary 7.5** The operator  $D + C\overline{K}_\lambda - C_\lambda B$  with domain  $\mathcal{D}(D) = H^2(\partial\Omega)$  is closed in  $Y$  and thus coincides with  $\overline{M}_\lambda$ .

As a result, the operator  $\mathcal{A}_0$  is closable and we now give an explicit description of its closure  $\mathcal{A}$ .

**Theorem 7.6** The operator  $\mathcal{A}_0$  is closable in  $\mathcal{X}$  and its closure  $\mathcal{A}$  acts according to

$$\mathcal{A} \begin{pmatrix} x + K_\lambda y - (A_1 - \lambda I)^{-1} B y \\ y \end{pmatrix} = \begin{pmatrix} A x + \lambda K_\lambda y - \lambda (A_1 - \lambda I)^{-1} B y \\ C x + \overline{M}_\lambda y \end{pmatrix}$$

on the domain

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{pmatrix} x + K_\lambda y - (A_1 - \lambda I)^{-1} B y \\ y \end{pmatrix} \mid x \in \mathcal{D}(A_1), y \in \mathcal{D}(D) \right\}.$$

### 7.4 Spectrum of the operator $\mathcal{A}$

In this subsection we use the results of Section 4 to study the spectrum of the operator  $\mathcal{A}$ .

Recall that the operator  $A_1$  has discrete spectrum by Proposition 7.1. If we fix  $\lambda_0 \in \rho(A_1)$ , then the resolvent  $(A_1 - \lambda_0 I)^{-1}$  maps  $X$  boundedly into  $H^2(\Omega)$  and  $C$  maps  $H^2(\Omega)$  into  $L_2(\partial\Omega)$  compactly, so  $C$  is  $A_1$ -compact. Theorem 4.2 implies that the essential spectrum of  $\mathcal{A}$  coincides with the essential spectrum of  $\overline{M}_{\lambda_0}$ .

On the other hand,  $D$  has discrete spectrum (see Subsection 7.2) and  $\overline{M}_{\lambda_0} = D + C\overline{K}_{\lambda_0} - C_{\lambda_0} B$  is a bounded perturbation of  $D$ . Therefore the essential spectrum of  $\overline{M}_{\lambda_0}$  is empty, and we have proved the following proposition.



**Proposition 7.7** *Under the assumptions of Subsection 7.1 the operator  $\mathcal{A}$  has discrete spectrum.*

More can be said about the distribution of the eigenvalues of  $\mathcal{A}$  under additional assumptions on the operators  $A$  and  $D$ . For example, suppose that for the principal symbol  $A_0(x, \xi)$  of  $A$  given by (7.2) there exists  $\theta \in [0, 2\pi)$  such that

$$\arg A_0(x, \xi) \neq \theta \quad (7.7)$$

for all  $x \in \overline{\Omega}$  and all vectors  $\xi \in \mathbb{R}^n \setminus \{0\}$ . Then the problem  $\{A - \lambda I, \mathbf{B}_1\}$  is elliptic with parameter  $\lambda = re^{i\theta}$ ,  $r > 0$  (see [1], [2, Chapter I]). Moreover,  $A_1$  has the following property.

**Proposition 7.8** (See [1, Theorem 2.1]) *If (7.7) is satisfied then  $R_\theta := \{re^{i\theta} \mid r > 0\}$  is a ray of minimal growth of the resolvent of  $A_1$ , i.e., there exist positive numbers  $r_\theta$  and  $c_\theta$  such that, for all  $r > r_\theta$ ,  $\lambda = re^{i\theta} \in \rho(A_1)$  and*

$$\|(A_1 - re^{i\theta}I)^{-1}\|_{L_2(\Omega)} \leq \frac{c_\theta}{r}.$$

**Lemma 7.9** *If (7.7) holds, then  $C\overline{K}_\lambda$  is uniformly bounded on the set  $R_\theta^0 := \{\lambda = re^{i\theta} \mid r \geq r_\theta\}$ , with  $r_\theta$  as in Proposition 7.8.*

**Proof.** Observe first that by definition  $R_\theta^0$  belongs to the resolvent set of  $A_1$  so that  $K_\lambda$  is well defined for  $\lambda \in R_\theta^0$ . In view of relation (7.6) (recall the proof of Lemma 7.4) it suffices to prove that the mapping  $K_\lambda : L_2(\partial\Omega) \rightarrow H^{3/2}(\Omega)$  is uniformly bounded in  $\lambda \in R_\theta^0$ .

Assume therefore that  $g \in H^{\ell-3/2}$  for some  $\ell \geq 3/2$  and put  $u = \overline{K}_\lambda g$  for  $\lambda \in R_\theta^0$ . Then  $u$  solves the problem

$$Au - \lambda u = 0, \quad B_1 u = g,$$

and [2, Theorem 4.1] implies that there exists a constant  $C > 0$  such that the inequality

$$\|u\|_{H^\ell(\Omega)} + |\lambda|^\ell \|u\|_{L_2(\Omega)} \leq C \left( \|g\|_{H^{\ell-3/2}(\partial\Omega)} + |\lambda|^{\ell-3/2} \|g\|_{L_2(\partial\Omega)} \right)$$

holds for all  $g \in H^{\ell-3/2}$  and all  $\lambda \in R_\theta^0$ . Taking  $\ell = 3/2$  we arrive at the desired conclusion.<sup>1</sup>  $\square$

Assume now that with the same value of  $\theta \in [0, 2\pi)$  as in (7.7) we have

$$\arg D_0(y, \eta) \neq \theta \quad (7.8)$$

for all  $y \in \partial\Omega$  and all nonzero  $\eta$  in the tangent space  $T_{\partial\Omega}(y)$  to  $\partial\Omega$  at the point  $y$ . Then also the ray  $R_\theta$  is a ray of minimal growth for the resolvent of  $D$ , and combination of the above results leads to the following theorem.

**Theorem 7.10** *In addition to the assumptions at the beginning of Subsection 7.1, suppose that conditions (7.7) and (7.8) are satisfied for the same  $\theta \in [0, 2\pi)$ . Then all sufficiently large  $\lambda \in R_\theta$  belong to the resolvent set of the operator  $\mathcal{A}$ .*

**Proof.** In view of the Frobenius–Schur factorization it suffices to show that the operator  $\overline{M}_\lambda - \lambda I$  is boundedly invertible for all sufficiently large  $\lambda \in R_\theta^0$ . We find that

$$\overline{M}_\lambda - \lambda I = (D - \lambda I)(I + (D - \lambda I)^{-1}[C\overline{K}_\lambda - C_\lambda B]).$$

Since the ray  $R_\theta$  is a ray of minimal growth for the resolvent of  $D$  and the operators  $C\overline{K}_\lambda$  and  $C_\lambda B$  are bounded in  $Y$  uniformly in  $\lambda \in R_\theta^0$  by Lemma 7.9 and inequality (5.7), the operator

$$I + (D - \lambda I)^{-1}[C\overline{K}_\lambda - C_\lambda B]$$

is boundedly invertible for all sufficiently large  $\lambda \in \rho(D) \cap R_\theta^0$ , and hence the same holds for  $\overline{M}_\lambda - \lambda I$ .  $\square$

<sup>1</sup> Strictly speaking, the above inequality is established in [2] only for integer  $\ell$ . However, by standard transposition and interpolation arguments from [20] it can be extended to all real  $\ell$ .

### 7.5 Semigroup generation

Assume now that both  $A$  and  $D$  generate holomorphic operator semigroups in  $X$  and  $Y$ , respectively. A sufficient (and in fact necessary) condition for this is that the principal symbols  $A_0(x, \xi)$  and  $D_0(y, \eta)$  of  $A$  and  $D$  are sectorial, i.e., that there exists a  $\theta_0 \in (0, \pi/2)$  such that

$$|\arg A_0(x, \xi)| < \theta_0, \quad |\arg D_0(y, \eta)| \leq \theta_0$$

for all  $x \in \Omega$ ,  $y \in \partial\Omega$ , and all nonzero  $\xi \in \mathbb{R}^n$  and  $\eta \in T_{\partial\Omega}(y)$ , see [21, Theorem 3.1.3]. In fact, under this condition any ray  $R_\theta$  with  $|\theta| \leq \pi - \theta_0$  is a ray of minimal growth for the resolvents of  $A_1$  and  $D$ , and hence  $A_1$  and  $D$  generate holomorphic operator semigroups in  $X$  and  $Y$ , respectively. Then  $\overline{M}_\lambda$ ,  $\lambda \in \rho(A_1)$ , is also a generator of a holomorphic semigroup in  $Y$ . Applying Theorem 5.1 and Proposition 5.3, we arrive at the following statement:

**Theorem 7.11** *In addition to the assumptions at the beginning of Subsection 7.1, suppose that  $A_1$  and  $D$  generate holomorphic operator semigroups in  $X$  and  $Y$ , respectively. Then  $\mathcal{A}$  is the generator of a holomorphic semigroup in  $\mathcal{X}$ .*

This result also follows from the observation that, under the assumptions of Theorem 7.10, all  $R_\theta$  with  $|\theta| < \pi - \theta_0$  are rays of minimal growth for the resolvent of  $\mathcal{A}$ .

## 8 Parabolic problems with boundary feedback

Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , be a bounded smooth domain as in Section 7. We consider the following initial value problem:

$$\begin{aligned} \partial_t u(x, t) &= \Delta_\Omega u(x, t) + \int_{\partial\Omega} l(x, z) u(z, t) dz, & x \in \Omega, \\ \partial_t u(z, t) &= \Delta_{\partial\Omega} u(z, t) + \int_{\Omega} k(x, z) u(x, t) dx, & z \in \partial\Omega, \\ u(x, 0) &= f_0(x), & x \in \Omega, \\ u(z, 0) &= g_0(z), & z \in \partial\Omega. \end{aligned} \tag{8.1}$$

Here  $k, l \in L_2(\Omega \times \partial\Omega)$ ,  $\Delta_\Omega$  is the Laplacian in  $\Omega$  and  $\Delta_{\partial\Omega}$  is the Laplace–Beltrami operator on  $\partial\Omega$ . We mention that more general uniformly elliptic operators, as in Section 7, could also be chosen. The initial data are supposed to satisfy the conditions  $f_0 \in L_2(\Omega)$ ,  $g_0 \in L_2(\partial\Omega)$ .

Using ideas from [9], we can write this system abstractly as follows. Introduce  $X = L_2(\Omega)$ ,  $Y = L_2(\partial\Omega)$ ,  $Z = H^{-3/2}(\partial\Omega)$ ,  $A = \Delta_\Omega$ ,  $D = \Delta_{\partial\Omega}$  with  $\mathcal{D}(A) = H^2(\Omega)$  and  $\mathcal{D}(D) = H^2(\partial\Omega)$ ,  $(Bg)(x) = \int_{\partial\Omega} l(x, z) g(z) dz$ ,  $x \in \Omega$ ,  $(Cf)(z) = \int_{\Omega} k(x, z) f(x) dx$ ,  $z \in \partial\Omega$ ,  $\Gamma_X = \gamma$ , the trace operator, and  $\Gamma_Y$  being the embedding of  $L_2(\partial\Omega)$  into  $H^{-3/2}(\partial\Omega)$ . Then the system (8.1) becomes

$$\mathbf{u}'(t) = \mathcal{A}_0 \mathbf{u}(t), \quad \mathbf{u}(0) = \begin{pmatrix} f_0 \\ g_0 \end{pmatrix}$$

with

$$\mathbf{u}(t) = \begin{pmatrix} u(\cdot, t) \\ \gamma u(\cdot, t) \end{pmatrix} \in \mathcal{X} = X \times Y$$

and

$$\begin{aligned} \mathcal{D}(\mathcal{A}_0) &:= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x \in \mathcal{D}(A), y \in \mathcal{D}(D), \Gamma_X x = \Gamma_Y y \right\}, \\ \mathcal{A}_0 \begin{pmatrix} x \\ y \end{pmatrix} &:= \begin{pmatrix} Ax + By \\ Cx + Dy \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{D}(\mathcal{A}_0). \end{aligned}$$

This operator is a bounded perturbation of the operator which was discussed in a more general situation in Section 7. Therefore we omit the details of verification of assumptions (i)–(viii). We mention only that the closure  $\mathcal{A}$  of

the operator  $\mathcal{A}_0$  generates a holomorphic semigroup  $(\mathcal{T}(t))_{t \geq 0}$ , and hence the above problem is well-posed. The location of the spectrum of  $\mathcal{A}$  determines asymptotic properties of the solutions to this problem, as the following result shows.

**Theorem 8.1** *Under the assumptions at the beginning of this section there exist subspaces  $\mathcal{X}_S$ ,  $\mathcal{X}_U$  and  $\mathcal{X}_C$  which are invariant under the semigroup  $(\mathcal{T}(t))_{t \geq 0}$  and such that  $\mathcal{X} = \mathcal{X}_S \oplus \mathcal{X}_C \oplus \mathcal{X}_U$ ,  $\dim \mathcal{X}_C < \infty$ ,  $\dim \mathcal{X}_U < \infty$ , and*

- (i) *the semigroup  $\mathcal{T}_S(t) = \mathcal{T}(t)|_{\mathcal{X}_S}$  is uniformly exponentially stable,*
- (ii) *the semigroup  $\mathcal{T}_U(t) = \mathcal{T}(t)|_{\mathcal{X}_U}$  is invertible and the semigroup  $\mathcal{T}_U^{-1}(t)$  is uniformly exponentially stable,*
- (iii) *the semigroup  $\mathcal{T}_C(t) = \mathcal{T}(t)|_{\mathcal{X}_C}$  can be extended in a natural way to a group which is polynomially bounded in both time directions and hence has growth bound 0 in both directions.*

*Further, if  $\sigma(\mathcal{A}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0\}$ , then  $\mathcal{X}_C = \mathcal{X}_U = \{0\}$  and hence the semigroup  $(\mathcal{T}(t))_{t \geq 0}$  is uniformly exponentially stable.*

The subspaces  $\mathcal{X}_S$ ,  $\mathcal{X}_U$  and  $\mathcal{X}_C$  are usually referred to as the corresponding *stable*, *unstable* and *centre* manifolds.

**Proof.** The theorem is an immediate consequence of the fact that  $\mathcal{A}$  generates a compact holomorphic semigroup by [13, Theorem II.4.29]. Define the sets

$$\begin{aligned}\Sigma_S &:= \{\lambda \in \sigma(\mathcal{A}) : \operatorname{Re} \lambda < 0\}, \\ \Sigma_C &:= \{\lambda \in \sigma(\mathcal{A}) : \operatorname{Re} \lambda = 0\}, \\ \Sigma_U &:= \{\lambda \in \sigma(\mathcal{A}) : \operatorname{Re} \lambda > 0\},\end{aligned}$$

and denote the corresponding spectral subspaces by  $\mathcal{X}_S$ ,  $\mathcal{X}_C$ , and  $\mathcal{X}_U$ . It follows from sectoriality and compactness of the resolvent of  $\mathcal{A}$  that  $\dim \mathcal{X}_C < \infty$  and  $\dim \mathcal{X}_U < \infty$ . Further, by [13, Proposition IV.1.16],  $\mathcal{X} = \mathcal{X}_S \oplus \mathcal{X}_C \oplus \mathcal{X}_U$ , and these subspaces are invariant under the semigroup  $(\mathcal{T}(t))_{t \geq 0}$ .

Since  $(\mathcal{T}_S(t))_{t \geq 0}$  is a compact semigroup in the Banach space  $\mathcal{X}_S$ , and its generator  $\mathcal{A}_S := \mathcal{A}|_{\mathcal{X}_S}$  has spectrum  $\sigma(\mathcal{A}_S) = \Sigma_S$ , it is uniformly exponentially stable. The other two statements follow from well-known results on matrix exponentials, since the groups  $(\mathcal{T}_C(t))_{t \in \mathbb{R}}$  and  $(\mathcal{T}_U(t))_{t \in \mathbb{R}}$  are exponentials of the matrices  $\mathcal{A}_C := \mathcal{A}|_{\mathcal{X}_C}$  and  $\mathcal{A}_U := \mathcal{A}|_{\mathcal{X}_U}$ , respectively. Further,  $\sigma(\mathcal{A}_C) = \Sigma_C$  and  $\sigma(\mathcal{A}_U) = \Sigma_U$ .  $\square$

## 9 Further examples

### 9.1 Delay differential equations

We shall use the notation  $I_{h,k}$  for the interval with ends  $-h$  and  $k$ , an endpoint being included if and only if it is finite. For a finite or infinite (nonnegative) delay  $h$ , and for a function  $u$  defined on  $I_{h,\infty}$  with values in a Banach space  $Y$ , we define the *history function*  $u_t : I_{h,0} \rightarrow Y$  by  $u_t(s) := u(t+s)$ . In the following we consider the problem

$$\begin{cases} u'(t) = Cu_t + Du(t), & t \geq 0, \\ u_0 = x \in L_p(I_{h,0}, Y) =: X, \\ u(0) = y \in Y, \end{cases} \quad (9.1)$$

where  $p \geq 1$ ,  $C$  is a bounded linear operator from  $W_p^1(I_{h,0}, Y)$  into  $Y$  and  $D$  is a closed operator in  $Y$  generating a strongly continuous semigroup. For example, if  $h = 1$  and  $C = \delta_{-1}$  we obtain the problem

$$\begin{cases} u'(t) = u(t-1) + Du(t), & t \geq 0, \\ u|_{[-1,0]} = x \in L_p([-1,0], Y), \\ u(0) = y \in Y. \end{cases} \quad (9.2)$$

It is well-known (see e.g., [4, 5, 6]) that problem (9.1) is equivalent to an abstract Cauchy problem in  $\mathcal{X} := X \times Y$  with the vector function  $\mathbf{v}(t) := \begin{pmatrix} u_t \\ u(t) \end{pmatrix}$ :

$$\mathbf{v}'(t) = \mathcal{A} \mathbf{v}(t), \quad t \geq 0, \quad \mathbf{v}(0) = \begin{pmatrix} x \\ y \end{pmatrix} \quad (9.3)$$

where

$$\mathcal{A} := \begin{pmatrix} \frac{d}{ds} & 0 \\ C & D \end{pmatrix} \quad (9.4)$$

with domain

$$\mathcal{D}(\mathcal{A}) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in W_p^1(I_{h,0}, Y) \times \mathcal{D}(D) : x(0) = y \right\}. \quad (9.5)$$

Now it is easy to check that this operator is of the matrix form of Section 3 with  $Z := Z_1 := Y$ ,  $\Gamma_Y := I$ ,  $\Gamma_X x := x(0)$  with  $\mathcal{D}(\Gamma_X) := W_p^1(I_{h,0}, Y) \subset X$ ,  $A = \frac{d}{ds}$  with  $\mathcal{D}(A) = W_p^1(I_{h,0}, Y)$  and  $B = 0$ .

The operator  $A_1 = \frac{d}{ds}$  with

$$\mathcal{D}(A_1) = \{x \in W_p^1(I_{h,0}, Y) : x(0) = 0\}$$

is the generator of the left shift semigroup

$$(T(t)y)(s) := \begin{cases} y(t+s), & t+s < 0, \\ 0, & t+s \geq 0, \end{cases}$$

which is nilpotent if  $h < \infty$ . We obtain

$$\mathcal{N}(A - \lambda I) = \{e^{\lambda \cdot} y : y \in Y\}$$

for  $\lambda \in \mathbb{C}$  if  $h < \infty$  and for all  $\lambda$  with  $\operatorname{Re} \lambda > 0$  if  $h = \infty$ .

In applications the operator  $C$  often has a representation

$$Cx := \int_{-h}^0 d\eta(s) x(s), \quad (9.6)$$

where  $\eta \in \operatorname{BV}(I_{h,0}, \mathcal{L}(Y))$  is a given function. Then  $K_\lambda y = e^{\lambda \cdot} y$  and  $CK_\lambda y = \int_{-h}^0 e^{\lambda s} d\eta(s) y$ .

Under these assumptions, we can now verify the conditions (i)–(viii). Since  $A$  is closed and  $\mathcal{D}(A) = \mathcal{D}(\Gamma_X)$ , assumptions (i) and (ii) are satisfied. Assumption (iii) follows from the fact that  $A_1$  generates a strongly continuous semigroup. Since  $C$  is of the form (9.6), (iv) follows and (v) is a consequence of  $K_\lambda y = e^{\lambda \cdot} y$ . Assumption (vi) was made earlier and (vii) and (viii) follow trivially. Further, in this case, the operator  $\mathcal{A}$  is already closed, see [5, Lemma 2.1].

For the spectral characterization we again make a distinction between the finite and infinite delay cases.

**Theorem 9.1** *With the operator  $\mathcal{A}$  as in (9.4), (9.5), if  $h < \infty$  then*

$$\lambda \in \rho(\mathcal{A}) \iff \lambda \in \rho(D + CK_\lambda);$$

*if  $h = \infty$ , then this equivalence holds only for  $\operatorname{Re} \lambda > 0$ . In both cases, the operator  $\mathcal{A}$  generates a strongly continuous semigroup.*

**Proof.** The resolvent characterization is a direct consequence of Lemma 3.1. The generation property was shown in [5, Examples 3.4] and [22] via the fact that in this case  $\mathcal{A} = \hat{\mathcal{A}} + \mathcal{C}$  with

$$\hat{\mathcal{A}} := \begin{pmatrix} \frac{d}{ds} & 0 \\ 0 & D \end{pmatrix}, \quad \mathcal{D}(\hat{\mathcal{A}}) := \mathcal{D}(\mathcal{A}) \quad \text{and} \quad \mathcal{C} := \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}.$$

Now the perturbation theorem of Miyadera–Voigt, see [13, Theorem III.3.14], can be applied. For a thorough treatment we refer to [6].  $\square$

For the results concerning the essential spectrum in Section 4 we assumed  $C$  to be  $A_1$ -compact. For delay differential equations this is satisfied if and only if  $Y$  is finite dimensional. Thus we obtain the following well-known result for finite dimensional  $Y$  (recall that in this case  $\sigma_{ess}(D) = \emptyset$ ).

**Proposition 9.2** *Assume that  $\dim Y < \infty$  and that  $h < \infty$ . Then  $\sigma_{ess}(\mathcal{A}) = \emptyset$ .*

## 9.2 Abstract observation systems

We consider the abstract observation system

$$\begin{cases} y'(t) = Dy(t), & y(0) = y_0, \\ z(t) = Gy(t), \end{cases}$$

for functions  $y, z$  with values in Banach spaces  $Y, Z$ , respectively. Here  $D$  is a linear operator from  $\mathcal{D}(D) \subset Y$  into  $Y$ ,  $D$  generates a strongly continuous semigroup  $T(t)$  in  $Y$ , and  $G$  is a bounded linear operator from  $\mathcal{D}(D)$  into  $Z$ . Here  $\mathcal{D}(D)$  is endowed with the graph norm of  $D$ . The operator  $D$  can be considered to govern the free problem, which is well-posed since  $D$  generates a strongly continuous semigroup, and  $G$  is the observation operator. Thus the system represents a kind of a black box, where the function  $y$  describes the state of the system and the function  $z$  describes what we can observe.

Let  $1 \leq p < \infty$ . Recall (cf. [28, Definition 6.1]) that in control theory the observation operator  $G$  is called *p-admissible* if there exist  $t_0, M > 0$  such that

$$\int_0^{t_0} \|GT(t)y\|_Z^p dt \leq M \|y\|_Y^p, \quad y \in \mathcal{D}(D)$$

for all  $y \in \mathcal{D}(D)$ . We shall use the following characterization of *p-admissibility* (see [11, Theorem 2(b)]):

*The observation operator  $G \in \mathcal{L}(\mathcal{D}(D), Z)$  is p-admissible if and only if there exists a  $t_0 > 0$  such that the operator matrix*

$$\mathcal{A} := \begin{pmatrix} -\frac{d}{ds} & 0 \\ 0 & D \end{pmatrix} \quad (9.7)$$

with

$$\mathcal{D}(\mathcal{A}) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in W_p^1([0, t_0], Z) \times \mathcal{D}(D) : x(0) = Gy \right\} \quad (9.8)$$

generates a strongly continuous semigroup in  $\mathcal{X} := L_p([0, t_0], Z) \times Y$ .

This operator  $\mathcal{A}$  can be treated within the framework of Section 2. To this end we choose  $X = L_p([0, t_0], Z)$ ,  $A := -\frac{d}{ds}$  with  $\mathcal{D}(A) = W_p^1([0, t_0], Z)$ ,  $\Gamma_X : \mathcal{D}(A) \rightarrow Z$  with  $\Gamma_X(x) := x(0)$ ,  $\Gamma_Y := G$ , and  $B = C = 0$ . It follows that assumptions (i), (ii), (iv) are satisfied and for assumption (iii) we obtain  $A_1 := -\frac{d}{ds}$  with  $\mathcal{D}(A_1) := \{x \in W_p^1([0, t_0], Z) : x(0) = 0\}$ , which is the generator of the (nilpotent) right shift semigroup. Hence,  $\rho(A_1) = \mathbb{C}$ . Following the identification, we see that  $\mathcal{N}(A - \lambda I) = \{ze^{-\lambda \cdot} : z \in Z\}$  and hence  $Z_1 := Z$ .

The operator  $K_\lambda : Z \rightarrow \mathcal{N}(A - \lambda I)$  can be defined, analogously to Subsection 9.1, via

$$K_\lambda(z) := ze^{-\lambda \cdot}.$$

Thus condition (v) is satisfied for all  $\lambda \in \mathbb{C}$ . Finally, the conditions (vi) and (vii) were assumed at the beginning of this section, and (viii) is trivially satisfied.

Thus, the results of Section 2 can be applied to the operator  $\mathcal{A}$ . In particular, it can be represented for all  $\lambda \in \mathbb{C}$  via

$$\mathcal{A} - \lambda \mathcal{I} = \begin{pmatrix} A_1 - \lambda I & 0 \\ 0 & D - \lambda I \end{pmatrix} \begin{pmatrix} I & -K_\lambda \Gamma_Y \\ 0 & I \end{pmatrix}, \quad (9.9)$$

and since  $0 \in \rho(A_1)$ , we also obtain

$$\mathcal{A} = \begin{pmatrix} A_1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & -K_0\Gamma_Y \\ 0 & I \end{pmatrix}.$$

Thus the operator  $\mathcal{A}$  is a multiplicative perturbation of a semigroup generator which corresponds to the system without observation (i.e., with  $G = 0$ ).

This fact opens up possibilities of using the known theory of such perturbations, cf. [10], to characterize and understand  $p$ -admissibility of observation operators for generators  $\mathcal{A}$ . This, however, lies beyond the scope of the present paper.

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## References

- [1] S. Agmon, On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems, *Comm. Pure Appl. Math.* **15**, 119–147 (1962).
- [2] M. S. Agranovich and M. I. Vishik, Elliptic problems with parameter and parabolic problems of a general form, *Uspekhi Matem. Nauk* **19**, 53–161 (1964) (in Russian).
- [3] F. V. Atkinson, H. Langer, R. Mennicken, and A. A. Shkalikov, The essential spectrum of some matrix operators, *Math. Nachr.* **167**, 5–20 (1994).
- [4] A. Bátkai, Hyperbolicity of linear partial differential equations with delay, *Integral Equations Operator Theory* **44**, 383–396 (2002).
- [5] A. Bátkai and S. Piazzera, Semigroups and linear partial differential equations with delay, *J. Math. Anal. Appl.* **264**, 1–20 (2001).
- [6] A. Bátkai and S. Piazzera, *Semigroups for Delay Equations in  $L^p$  Phase Spaces* (A. K. Peters, Wellesley, Ma, 2005).
- [7] P. Binding and R. Hryniv, Relative boundedness and relative compactness for linear operators in Banach spaces, *Proc. Amer. Math. Soc.* **128**, 2287–2290 (2000).
- [8] P. Binding, R. Hryniv, H. Langer, and B. Najman, Elliptic eigenvalue problems with eigenparameter dependent boundary conditions, *J. Differential Equations* **174**, 30–54 (2001).
- [9] V. Casarino, K. J. Engel, R. Nagel, and G. Nickel, A semigroup approach to boundary feedback systems, *Integral Equations Operator Theory* **47**, 289–306 (2003).
- [10] W. Desch and W. Schappacher, Some generation results for perturbed semigroups. Semigroup theory and applications, in: *Lecture Notes in Pure and Applied Mathematics Vol. 116* (Dekker, New York, 1989), pp. 125–152.
- [11] K. J. Engel, On the characterization of admissible control- and observation operators, *Systems Control Lett.* **34**, 225–227 (1998).
- [12] K. J. Engel, Spectral theory and generator property for one-sided coupled operator matrices, *Semigroup Forum* **58**, 267–295 (1999).
- [13] K. J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations* (Springer-Verlag, New York, 2000).
- [14] J. Ercolano and M. Schechter, Spectral theory for operators generated by elliptic boundary problems with eigenvalue parameter in boundary conditions, I. *Comm. Pure Appl. Math.* **18**, 83–105 (1965).
- [15] I. C. Gohberg and M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, *Translations of Mathematical Monographs Vol. 18* (American Mathematical Society, Providence, RI, 1969).
- [16] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, *Monographs and Studies in Mathematics Vol. 24* (Pitman, Boston, Ma, 1985).
- [17] T. Kato, *Perturbation Theory for Linear Operators* (Springer-Verlag, Berlin–New York, 1976).
- [18] M. Langer, Eigenvalues of a  $\lambda$ -rational Sturm–Liouville problem, *Math. Nachr.* **210**, 163–176 (2000).
- [19] H. Langer and C. Tretter, Diagonalization of certain block operator matrices and applications to Dirac operators, *Oper. Theory Adv. Appl.* **122**, 331–358 (2001).

- [20] J.-L. Lions and E. Magenes, Non-Homogeneous Boundary Value Problems and Applications Vol. 1 (Springer-Verlag, Berlin – Heidelberg, 1972).
- [21] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems (Birkhäuser, Basel, 1995).
- [22] L. Maniar and J. Voigt, Linear delay equations in the  $L^p$  context, in: Evolution Equations, Lecture Notes in Applied Mathematics Vol. 234 (Dekker, New York, 2003), pp. 319–330.
- [23] R. Nagel, Towards a “matrix theory” for unbounded operator matrices, Math. Z. **201**, 57–68 (1989).
- [24] R. Nagel, The spectrum of unbounded operator matrices with non-diagonal domain, J. Funct. Anal. **89**, 291–302 (1990).
- [25] A. A. Shkalikov, On the essential spectrum of matrix operators, Math. Notes **58**, 945–949 (1995) (in Russian).
- [26] K. Taira, A. Favini, and S. Romanelli, Feller semigroups and degenerate elliptic operators with Wentzell boundary conditions, Studia Math. **145**, 17–53 (2001).
- [27] M. Taylor, Partial Differential Equations, Vol. 1: Basic Theory (Springer-Verlag, New York, 1996).
- [28] G. Weiss, Admissible observation operators for linear semigroups, Israel J. Math. **65**, 17–43 (1989).
- [29] J. Wu, Theory and Applications of Partial Functional Differential Equations, Applied Mathematical Sciences Vol. 119 (Springer, 1996).